RELATIVE PROJECTIVE COVERS AND THE BRAUER CONSTRUCTION OVER FINITE GROUP ALGEBRAS (Cohomology Theory of Finite Groups and Related Topics)

OKUYAMA, TETSURO

数理解析研究所講究録
第 1784 巻 2012 年 77-95

RELATIVE PROJECTIVE COVERS AND THE BRAUER CONSTRUCTION OVER FINITE GROUP ALGEBRAS

北海道教育大学旭川校 奥山 哲郎 (TETSURO OKUYAMA)
HOKKAIDO UNIVERSITY OF EDUCATION, ASAHIKAWA CAMPUS

Some properties of relative projective covers of modules in the modular representation theory of finite groups will be discussed. Especially, we study effects of the Brauer constructions for relative projective covers of p-permutation modules. We also discuss some use of our results to investigate derived equivalences in the principal block algebras of finite groups with Sylow p-subgroup isomorphic to $M_{n+1}(p)$, p odd.

In my lecture, we only talked on the sections 1 and 2 below. We include the sections 3 and 4 which provide some results for proofs of theorems in sections 2. Section 5 is also included to give another examples with finite groups with Sylow p-subgroups $M_{n+1}(p)$.

Let k be an algebraically closed field of characteristic $p > 0$.

1. $M_{n+1}(p)$

Let p be odd and n be an integer with $n \geqq 2$. The p-group $M_{n+1}(p) = P$ of order p^{n+1} is presented by

$$M_{n+1}(p) = P \;;= \langle\, x, y \mid y^{p^n} = 1 = x^p,\ xyx^{-1} = y^{1+p^{n-1}} \,\rangle$$

P has a unique maximal elementary abelian p-subgroup $\langle\, x,\ y^{p^{n-1}} \,\rangle$. Set

$$Q = \langle\, y \,\rangle, \quad R = \langle\, x \,\rangle$$

We fix an integer $s \in \mathbb{Z}$ which has multiplicative order $p-1$ in the residue ring $\mathbb{Z}/p^n\mathbb{Z}$. Notice then that s has multiplicative order $p-1$ in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ also. P has an automorphism t_0 of order $p-1$ which sends $x \mapsto x,\ y \mapsto y^s$ so that we have a group .

$$P \rtimes \langle\, t_0 \,\rangle \;;= \langle\, x, y, t_0 \mid y^{p^n} = 1 = x^p,\ xyx^{-1} = y^{1+p^{n-1}},\ {t_0}^{p-1} = 1,\ {t_0}^{-1} x t_0 = x,\ {t_0}^{-1} y t_0 = y^s \,\rangle$$

In the following discussion, fix a positive divisor $e \geqq 2$ of $p-1$ and set $t = {t_0}^{\ell}$ where $\ell = \dfrac{p-1}{e}$. And set

$$H = P \rtimes \langle\, t \,\rangle \cong P \rtimes \mathbb{Z}_e$$

1.1. **Some Complexes of kH-modules.** kH has e simple modules $S(i)$, $i \in \mathbb{Z}/e\mathbb{Z}$ (all of dimension 1). We can name the simples so that the following facts hold.

$$\mathrm{Ext}^1_{kH}(S(i), S(i+1)) \neq 0, \quad S(0) = k_H$$

Let denote a projective cover of $S(i)$ by $P(i)$.

By a result of Okuyama and Sasaki [7], we have a (chain) complex $X^\bullet(1)$ of kH-modules

(1.1)

$$\begin{array}{l} X_1^\bullet(1)\ :\ \cdots \longrightarrow \quad S(1) \quad \longrightarrow P(1) \quad \to P(1)\oplus P(1) \to \Omega^{-2e}(S(1)) \to 0 \to \cdots \\ X_k^\bullet(1)\ :\ \cdots \to \Omega^{-2(k-1)e}(S(1)) \to P(1)\oplus P(1) \to P(1)\oplus P(1) \to \Omega^{-2ke}(S(1)) \to 0 \to \cdots \end{array}$$

satisfying $H_1(X_k^\bullet(1)) \cong H_2(X_k^\bullet(1)) = \begin{array}{c} S(2) \\ \vdots \\ S(-1) \\ S(0) \end{array}$ and $H_0(X_k^\bullet(1)) \cong H_3(X_k^\bullet(1)) = 0$ for each $1 \leqq k \leqq \ell$ where the last nonzero terms are in degree 0.

1.2. **Richard's Tilting.** Let A be an arbitraly symmetric algebra over k and $\{\ S(i)\ ;\ i \in I\ \}$ be the set of simple A-modules. Let $P(i)$ be a projective cover of $S(i)$.

Take a (nomempty proper) subset I_0 of I. For each $i \in I$, construct a complex $P^\bullet(i) \in C^b(P(A))$ of projective A-modules as follows.

$$\begin{array}{lllll} P^\bullet(j)\ : & \cdots \to 0 \to R(j) & \xrightarrow{\lambda_j} P(j) \to 0 \to \cdots & & j \notin I_0 \\ P^\bullet(i)\ : & \cdots \to 0 \to P(i) & \longrightarrow \quad 0 \ \to 0 \to \cdots & & i \in I_0 \end{array}$$

where for $j \notin I_0$, $R(j) \xrightarrow{\lambda_j} P(j)$ is a minimal one satisfying that

(1). $R(j)$ is a direct sum of $P(i)$, $i \in I_0$

(2). Composition factors of $\operatorname{Cok}\lambda_j$ are $S(k)$ for some $k \notin I_0$

Set

$$P^\bullet(I_0) = \oplus \sum\nolimits_{i\in I} P^\bullet(i)$$

Then in the homotopy category $K^b(P(A))$ of complexes of projective A-modules,

$P^\bullet(I_0)$ is a **tilting complex** for A

Set

$$B = \operatorname{End}_{K^b(P(A))}(P^\bullet(I_0))$$

so that B is a **derived equivalent** algebra to A. B is also a symmetric algebra and simple B-modules are also parametrized by the set I. Let $Q(i)$ be a projective indecomposable B-module corresponding to the summand $P^\bullet(i)$ of $P^\bullet(I_0)$. Let $T(i)$ be the simple B-module corresponding to $T(i)$. There is a (A,B)-bimodule $M(I_0)$ (with no bimodule projective summand) constructed from the complex $P^\bullet(I_0)$ satisfying the following.

(1). Both of ${}_AM(I_0)$ and $M(I_0)_B$ are projective and a functor

$$F_0 = F(I_0) : \text{mod-}A \to \text{mod-}B, \quad V \mapsto V \otimes_A M(I_0)$$

gives a **stable equivalence of Morita type** between mod-A and mod-B.

(2.1). For $j \notin I_0$, $F_0(S(j)) = T(j)$ for $j \notin I_0$.

(2.2). For $i \in I_0$, let $\operatorname{Soc}P(i) \subset W(i) \subset P(i)$ be the largest submodule of $P(i)$ such that all the composition factors of $W(i)/S(i)$ are $S(k)$ for some $k \notin I_0$. Then $F_0(P(i)/W(i)) = T(i)$. for $i \in I_0$.

We call the procedure above the **Richard's Tilting** with respect to the set $I_0 \subset I$. The functor F_0 given above is called the associated functor of the tiltng. The dual argument to the above discussion is also valid which we call the **dual Richard's Tilting.**

1.2.1. *Examples.* For kH, apply Richard's tiltings with repect to the set $\{ 1 \}$ twice. First do the Richard's tilting with respect to the set $\{ 1 \}$. And then for the resulting new algebra, do the Richard's tilting with respect to the set $\{ 1 \}$.

Let A_2 be the resulting algebra and let $S(i)_2$ (resp. $P(i)_2$) be a simple (resp. projective) A_2-module corresponding to $S(i)$, $i \in I$. Let $F^2 : \text{mod-}kH \to \text{mod-}A_2$ be the associated functor. Then by the existence of the complex $X_1^\bullet(1)$ in (1.1), we have

Lemma 1.1.

$$F^2(S(i)) = S(i)_2, \ i \neq 1, \qquad F^2(S(1)) = \Omega^{2e}(S(1)_2)$$

The existence of comlexes $X_k^\bullet(1)$ $(1 \leqq k \leqq \ell)$ implies the following. For each k with $1 \leqq k \leqq \ell$, do the Richard's tiltings with repect to the set $\{ 1 \}$ $\underline{2k}$ times.

Let A_{2k} be the resulting algebra and let $S(i)_{2k}$ (resp. $P(i)_{2k}$) be a simple (resp. projective) A_{2k}-module corresponding to $S(i)$, $i \in I$. Let $F^{2k} : \text{mod-}kH \to \text{mod-}A_{2k}$ be the associated functor. Then

Lemma 1.2.

$$F^{2k}(S(i)) = S(i)_{2k}, \ i \neq 1, \qquad F^{2k}(S(1)) = \Omega^{2ke}(S(1)_{2k})$$

The discussion above is valid for any fixed $i_0 \in I$.

Lemma 1.3. *Let $i_0 \in I$ and k be an integer with $1 \leqq k \leqq \ell$.*

(1) *There exists an algebra B derived equivalent to kH satisfying the following. Let $T(i)$, $i \in I$ be the set of simple B-modules and $F^* : \text{mod-}kH \to \text{mod-}B$ be the associated stable equivalence. Then*

$$F^*(S(i)) = T(i), \ i \neq i_0, \qquad F^*(S(i_0)) = \Omega^{2ke}(T(i_0))$$

(2) *There exists an algebra C derived equivalent to kH satisfying the following. Let $U(i)$, $i \in I$ be the set of simple C-modules and $F_* : \text{mod-}kH \to \text{mod-}C$ be the associated stable equivalence. Then*

$$F_*(S(i)) = U(i), \ i \neq i_0, \qquad F_*(S(i_0)) = \Omega^{-2ke}(U(i_0))$$

1.3. **Relative Projective Covers.** Set

$$K = R \times \langle\, t \,\rangle = \langle\, x \,\rangle \times \langle\, t \,\rangle \subset H$$

and

$$P_R(i) = (S(i) \downarrow_K) \uparrow^H = P_R(0) \otimes S(i)$$

Then we have a canonical surjection and a canonical injection

$$P_R(0) \xrightarrow{\mu} S(0) \to 0, \qquad 0 \to S(0) \xrightarrow{\nu} P_R(0)$$

μ is so called an **(relative)** R**-projective cover** of $S(0) = k_H$ and ν is an **(relative)** R**-injective hull** of $S(0) = k_H$.

For any kH-module V, an R-projective cover (R-injective hull) of V is obtained as a summand of the sequence obtained by tensoring with the above sequences. Let $\Omega_R(V)$ (resp. ${\Omega_R}^{-1}(V)$) be the kernel (resp. cokernel) of an R-projective cover (resp. R-injective hull) of V. In particular, we have the following short eaxact sequences,

$$0 \to \Omega_R(S(0)) \to P_R(0) \xrightarrow{\mu} S(0) \to 0, \qquad 0 \to S(1) \xrightarrow{\nu} P_R(0) \to {\Omega_R}^{-1}(S(0)) \to 0$$

The **heart** $H_R(0)$ of $P_R(S(0)) = P_R(k_H)$ is defined by

$$H_R(0) = \operatorname{Ker}\mu / \operatorname{Im}\nu$$

1.3.1. *Examples.* By a result of Okuyama and Sasaki [7], we have

$${\Omega_R}^2(S(0)) \cong \Omega^{-2(p-1)}(S(1))$$

Actually, we can show that an R-projective cover of $\Omega_R(S(0))$ has the form

$$0 \to \Omega^{-2(p-1)}(S(1)) \to P(1) \oplus P_R(1) \to \Omega_R(S(0)) \to 0$$

so that we have the complex $X_0^\bullet$ of kH-modules of the form

$$X_0^\bullet \ : \ \cdots \to 0 \to \Omega^{-2(p-1)}(S(1)) \to P(1) \oplus P_R(1) \to H_R(0) \to 0 \to \cdots \tag{1.2}$$

which satisfies that

$$H_1(X_0^\bullet) = S(0), \qquad H_2(X_0^\bullet) = 0 = H_0(X_0^\bullet)$$

where $H_R(0)$ is in degree 0 term. Set

$$F_*(1) = \Omega^{-1}\Omega_R(H_R(0))$$

Then by the sequence (1.2), we have the complex $X^\bullet$ of kH-modules of the form

$$\begin{aligned} &X^\bullet \ : \ \cdots \to 0 \to \Omega^{-2(p-1)}(S(1)) \to P(1) \oplus P(1) \to F_*(1) \to 0 \to \cdots \\ &\text{which satisfies that} \\ &H_1(X^\bullet) = S(0), \qquad H_2(X^\bullet) = 0 = H_0(X^\bullet) \quad \text{and} \\ &F_*(1) \subset P(-1) \oplus P(1) \end{aligned} \tag{1.3}$$

where $F_*(1)$ is in degree 0 term.

Assume that $\underline{e = 2}$. Do the Richard's tiltings with repect to the set $\{ \ 1 \ \}$ $\underline{p = 2k+1}$ times.

Let A_0 be the resulting algebra and let $S(i)_0$ (resp. $P(i)_0$) be a simple (resp. projective) A_0-module corresponding to $S(i)$, $i \in I$. Let F^0 : mod- kH $\to$ mod- A_0 be the associated functor. Then by the existence of complexes $X_k^\bullet(1)$ in (1.1) and $X^\bullet$ in (1.3), we have the following lemma.

Lemma 1.4. *Assume that* $e = 2$. *Then in the notatios above, we have*

$$F^0(S(0)) = S(0)_0, \ i \neq 1, \qquad F^0(F_*(1)) = S(1)_0$$

2. Example $SL(2,q)$

The example here is one discussed by Holloway-Koshitani-Kunugi [4].

Let q_1 be a prime power and p be an odd prime such that p divides q_1+1. Write $q_1+1 = p^{n-1}\ell'$, $(p,\ell')=1, n \geqq 2$. Set $q = q_1{}^p$. Then $q+1 = p^n\ell$ for some positive integer ℓ with $(p,\ell)=1$.
Set

$$G_0 = SL(2,q),\ C_0 = SL(2,q_1).\quad R = \mathcal{G}(GF(q)/GF(q_1)) = \langle\ x\ \rangle,\quad G = R \ltimes G_0$$

Let $B_0 = T_0 \ltimes U_0$ be a Borel subgroup of G_0 where $|T_0| = (q-1)$ and $|U_0| = q$. We have an R-invariant subgroup $F_0 \supset Z(G_0)$ of order $q+1$ such that $F_0 \cap C_0$ is of order q_1+1 and $B_0 \cap F_0 = Z(G_0)$.
Let $P_0 \subset F_0$ be a Sylow p-subgroup of G_0 and set $P = R \ltimes P_0$. We have that $P \cong M_{n+1}(p)$.
We use notations introduced in the beginning of the talk.

So $Q = P_0$. Set $H = N_G(Q) = N_G(P_0)$. Then $H/O_{p'}(H)$ is our H with $e=2$. Set $H_0 = N_{G_0}(P_0) = H \cap G_0$.

2.1. $B_0(kG_0)$. The principal block algebra $B_0(kG_0)$ of kG_0 has a cyclic defect group and is well understood. It is known that $B_0(kG_0)$ and the principal block $B_0(kH_0)$ are derived equivalent. A **two sided tilting complex** for $B_0(kG_0)$ and $B_0(kH_0)$ due to Rouquier is given as follows. Set

$$A = B_0(kG_0),\quad B = B_0(kH_0)$$

$B_0(kG_0)$ and $B_0(kH_0)$ have two simple modules

$$B_0(kG_0)\ :\ \phi_0 = k_{G_0},\quad \phi_1,\quad \dim_k \phi_1 = q-1$$

$$B_0(kH_0)\ :\ T_0 = k_{H_0},\quad T_1,\quad \dim_k T_1 = 1$$

$B = B_0(kH_0)$ is a symmetric Nakayama algebra of length p^n.

Let $P(\phi_i)$ $(i = 0,\ 1)$ be a projective cover of ϕ_i and $P(T_i)$ $(i = 0,\ 1)$ be a projective cover of T_i.

A is a (A,B)-bimodule (a (kG_0,kH_0)-bimodule). As usual, we can regard A as $k[G\times H]$-module. Let M_0 be a **Broué-Puig indecomposable** (A,B)**-summand** of A. As a $k[G\times H]$-module, M_0 is a **Scott module** with **vertex** $\Delta P_0 = \{\ (a,a)\ ;\ a \in P_0\ \} \subset G_0 \times H_0$. Actually, for the group $GL(2,q)$, $M_0 = A$. Notice also that ${}_AM_0$, M_B are both projective.

A functor

$$F : \text{mod-}A \to \text{mod-}B,\quad V \mapsto V \otimes_A M_0$$

gives a stable equivalence of Morita type between mod-A and mod-B.
We can see that a ΔP_0-projective cover of $k = k_{G\times H}$ has the form

$$M_0 \xrightarrow{\pi} k \to 0$$

and $\operatorname{Top}\operatorname{Ker}\pi = \phi_1^* \otimes_k T(1)$ where $\phi_1^* = \operatorname{Hom}_k(\phi_1,k)$ is a left kG_0-module. Let $P(\phi_1)^* \otimes_k P(T_1) \xrightarrow{\lambda} \operatorname{Ker}\pi \to 0$ be a projective cover of $\operatorname{Ker}\pi$ and consider the following complex $M^\bullet$ of (A,B)-bimodules.

$$M^\bullet\ :\ \cdots \to 0 \to P(\phi_1)^* \otimes_k P(T_1) \xrightarrow{\lambda} M_0 \to 0 \to \cdots \tag{2.1}$$

The complex $M^\bullet$ satisfies the following conditions.

$$M^\bullet \otimes_B M^{\bullet *} \cong A[0] \oplus Z^\bullet, \qquad M^{\bullet *} \otimes_A M^\bullet \cong B[0] \oplus W^\bullet$$

in $C^b(\text{mod-}A^{op} \otimes A)$ and $C^b(\text{mod-}B^{op} \otimes B)$, respectively where $Z^\bullet$ is a contractible complex of projective (A, A)-bimodules and $W^\bullet$ is a contractible complex of projective (B, B)-bimodules.

$$F(\phi_0) = T_0, \qquad F(\phi_1) = \phi_1 \otimes_A M_0 = \begin{matrix} T_1 \\ T_0 \\ \vdots \\ T_0 \\ T_1 \end{matrix} \quad \text{of length } p^n - 2$$

and

$$\phi_0 \otimes_A M^\bullet \ : \ \cdots \to 0 \to 0 \to T_0 \to 0 \to \cdots$$
$$\phi_1 \otimes_A M^\bullet \ : \ \cdots \to 0 \to P(T_1) \xrightarrow{\pi_1} F(T_1) \to 0 \to \cdots$$

where $P(T_1) \xrightarrow{\pi_1} F(T_1) \to 0$ is a projective cover of $F(T_1)$.

As a complex of projective B-modules, $M_B^\bullet$ is a complex obtained by the Richard's tilting for the algebra B with respect to the set $I_0 = 1 \subset I = \{0, 1\}$. We have

$$P(\phi_0) \otimes_A M^\bullet \ : \ \cdots \to 0 \to P(T_1) \to P(T_0) \to 0 \to \cdots$$
$$P(\phi_1) \otimes_A M^\bullet \ : \ \cdots \to 0 \to P(T_1) \to 0 \to 0 \to \cdots$$

and

$$M_B^\bullet \cong P^\bullet(0) \oplus (q-1)P^\bullet(1)$$

2.2. Let $\Gamma = (G_0 \times H_0)\Delta R \subset G \times H$. $M_0^x = M_0$ so that M_0 is a $k\Gamma$-module and has a vertex ΔP. $M = M_0 \uparrow^{G \times H}$ is a Broué-Puig indecomposable $(B_0(kG), B_0(kH))$-module. There exists a **p-permutation** $k\Gamma$-module X_0 with vertex ΔR such that $X_0 \downarrow_{G \times H} = P(\phi_1)^* \otimes_k P(T_1)$. So it is natural to ask whether we can construct a complex $X^\bullet$ of $k\Gamma$-modules of the form

$$X^\bullet \ : \ \cdots \to 0 \to X_0 \xrightarrow{\mu} M_0 \to 0 \to \cdots$$

such that $X^\bullet \downarrow_{G_0 \times H_0} \cong M^\bullet$. If such a complex exits, then $X^\bullet \uparrow^{G \times H}$ gives a twosided tilting complex for $B_0(kG)$ and $B_0(kH)$.

However, <u>we can no have such a complex</u>.

2.3. Recall that $C_0 = SL(2, q_1) = C_{G_0}(R)$ and $N_G(R) = R \times C_0$. The principal block algebra $B_0(kC_0)$ has a cyclic defect group $Q_0 = C_Q(R)$ and the structure of $B_0(kC_0)$ is described in the entiely same way as in $B_0(kG_0)$. $B_0(kC_0)$ and $B_0(kN_{C_0}(Q_0))$ have two simple modules

$$B_0(kC_0) \ : \ \theta_0 = k_{C_0}, \quad \theta_1, \quad \dim_k \theta_1 = q_1 - 1$$
$$B_0(kN_{C_0}(Q_0)) \ : \ T_0' = k_{N_{C_0}(Q_0)}, \quad T_1', \quad \dim_k T_1' = 1$$

$B_0(kC_0)$ and $B_0(kN_{C_0}(Q_0))$ are derived equivalent. Let N_0 be a Broué-Puig indecomposable module for them and let $N_0^\bullet$ be the twosided tilting complex for them so that $N_0^\bullet$ has the form

$$N_0^\bullet \ : \ \cdots \to 0 \to P(\theta_1)^* \otimes P(T_1') \to N_0 \to 0 \to \cdots$$

Using the isomorphism $(C_0 \times N_{C_0}(Q_0))\Delta(R)/\Delta R = C_0 \times N_{C_0}(Q_0)$, we can lift $N_0^\bullet$ to a twosided tilting complex $N^\bullet$ for $B_0(kN_G(R))$ and $B_0(N_H(R))$.

$$(2.2) \qquad N^\bullet \ : \ \cdots \to 0 \to Y \to N \to 0 \to \cdots$$

N is a Broué-Puig indecomposable $(B_0(kN_G(R)), B_0(kN_H(R)))$-module. The **Brauer costructions** for M with respect to ΔR is N. If we set $X = X_1 \uparrow^{G\times H}$, then $X(\Delta R) = Y$. And we can construct a complex of $(B_0(kG), B_0(kH))$-bimodules $X^\bullet$ of the form

$$(2.3) \qquad X^\bullet \ : \ \cdots \to 0 \to X \xrightarrow{\mu} M \to 0 \to \cdots$$

such that $X^\bullet(\Delta R) \cong N^\bullet$. $X^\bullet(\Delta R)$ satisfies the following conditions.

$$X^\bullet \otimes_{B_0(kH)} X^{\bullet *} \cong B_0(kG)[0] \oplus Z^\bullet, \qquad X^{\bullet *} \otimes_{B_0(kG)} X^\bullet \cong B_0(kH)[0] \oplus W^\bullet$$

in $C^b(\text{mod-}B_0(kG)^{op} \otimes B_0(kG))$ and $C^b(\text{mod-}B_0(kH)^{op} \otimes B_0(kH))$, respectively where $Z^\bullet$ is a complex of projective $(B_0(kG), B_0(kG))$-bimodules and $W^\bullet$ is a contractible complex of projective $(B_0(kH), B_0(kH))$-bimodules. A way of construction of $X^\bullet$ by $Y^\bullet$ is a (verry special type of) **gluing methods of Rouquier**.

If we take a suitable projective $(B_0(kG), B_0(kH))$-bimodule X' and a map $X' \xrightarrow{\nu} M$ such that

$$X \oplus X' \xrightarrow{\mu\oplus\nu} M \to 0 \text{ (exact)}$$

Then the complex

$$(2.4) \qquad X'^\bullet \ : \ \cdots \to 0 \to X \oplus X' \xrightarrow{\mu\oplus\nu} M \to 0 \to \cdots$$

has the same properties as for $X^\bullet$ where the complexes $Z^\bullet$ and $W^\bullet$ have homologies concentraited in degree 0. In particular, if we set

$$M_1 = \Omega^{-1}(\mathrm{Ker}(\mu \oplus \nu))$$

, then A functor

$$F_1 : \text{mod-}B_0(kG) \to \text{mod-}B_0(kH), \quad V \mapsto V \otimes_{B_0(kG)} M_1$$

gives a stable equivalence of Morita type between mod-$B_0(kG)$ and mod-$B_0(kH)$. We have the following lemma.

Lemma 2.1.

$$F_1(\varphi_0) = S(0), \qquad F_1(\varphi_1) = \Omega^{-1}\Omega_R(S(0))$$

Thus by Lemma 1.4, the following result follows.

Corollary 2.2 (Holloway-Koshitani-Kunugi [4]).
$B_0(kG)$ and $B_0(kH)$ are derived equivalent.

The procedure of Richard's tilting in the previous section implies that the resulting twosided tilting complex has the following form

$$\cdots \to 0 \to X_p \to X_{p-1} \to \cdots \to X_2 \to X_1 \oplus X_1' \to M \to 0 \to \cdots$$

The results in this section are obtained through the discussions with Koshitani and Kunugi.

3. Relative Projective Covers and Brauer Construction

3.1. Relative Projective Coveres. Let G be a finite group and $\mathfrak{X}$ be a nonempty family of subgroups of G. For a kG-module M, a short exact sequence $\mathbf{M}$; $0 \to N \to X \to M \to 0$ of kG-module is called $\mathfrak{X}$-projective cover of M if it satisfies

(1) X is $\mathfrak{X}$-projective,
(2) the sequence $\mathbf{M}$ is $\mathfrak{X}$-split.

For a kG-module M, a minimal $\mathfrak{X}$- projective cover of M exists and is uniquely determined up to isomorphism of exact sequences. An arbitraly $\mathfrak{X}$-projective cover contains a minimal one as a summand of exact sequences. If the above sequence $\mathbf{M}$ is minimal, then we denote N by $\Omega_{\mathfrak{X}}(M)$. M is $\mathfrak{X}$-projective if and only if $\Omega_{\mathfrak{X}}(M) = 0$. For kG-modules M and M', $\Omega_{\mathfrak{X}}(M \oplus M') = \Omega_{\mathfrak{X}}(M) \oplus \Omega_{\mathfrak{X}}(M)$.

Let H be a subgroup of G and set $\mathfrak{Y} = \mathfrak{X}^G \cap H = \{ A^g \cap H \; ; g \in G, \; A \in \mathfrak{X} \}$. Then the short exact sequence of kH-module $\mathbf{M} \downarrow_H$; $0 \to N \downarrow_H \to X \downarrow_H \to M \downarrow_H \to 0$ is a $\mathfrak{Y}$-projective presentation of a kH-module $M \downarrow_H$, not necessarily minimal even if $\mathbf{M}$ is minimal.

3.2. Brauer Construction.

3.3. Let G be a finite group and Q be a p-subgroup of G. Then a functor called the Brauer construction with respect to Q ;

$$-(Q) \; : \; mod(kG) \to mod(kN_G(Q)/Q)$$

is defined by

$$M(Q) = M^Q / (\sum_{R \subsetneq Q} \mathrm{Tr}_{R,Q}(M^R))$$

The canonical epimorphism from $M^Q \to M(Q)$ is denoted by Br_Q and is called the Brauer homomorphism with respect to Q.
If M and N are kG-modules and $f : M \to N$ is a kG-homomorphism, f induces a $kN_G(Q)/Q$-homomorphism $f(Q) : M(Q) \to N(Q)$. We denote $f(Q)$ by $\mathrm{Br}_Q(f)$. The Green correspondence with respect to $(G, N_G(Q), Q)$ gives a bijection between the set of isomorphism classes of indecomposable p-permutation kG-modules with vertex Q and the set of isomorphism classes of indecomposable projective $kN_G(Q)/Q$-modules. If X is an indecomposable p-permutation kG-modules with vertex Q, then the corresponding indecomposable projective $kN_G(Q)/Q$-module is the Brauer construction $X(Q)$.

Lemma 3.1. *Assume that $M \downarrow_Q$ is a permutation kQ-module. If $M(Q)$ has a projective $kN_G(Q)/Q$-summand U, then M has a Q-projective summand V with vertex Q such that $V(Q) = U$.*

Proof. $kN_G(Q)$-module $M \downarrow_{N_G(Q)}$ satisfies the assumption in the lemma for the group $N_G(Q)$ and a p-subgroup Q of $N_G(Q)$. Thus by a theorem of Burry-Carlson, we may assume that Q is normal in G. Let

$$X \xrightarrow{f} M \to 0 \qquad 0 \to M \xrightarrow{g} Y$$

be a Q-projective cover and a Q-injective hull of M, respectively. As $M \downarrow_Q$ is a permutation module, X and Y are Q-projective, p-permutation kG-modules. In particular,

$X(Q)$ and $Y(Q)$ are projective kG/Q-modules. As the sequence above are Q-split, we have exact sequences,

$$X(Q) \xrightarrow{f(Q)} M(Q) \to 0, \qquad 0 \to M(Q) \xrightarrow{g(Q)} Y(Q)$$

There exists a primitive idempotent $e \in kG$ such that $e[Q]kG \cong U$. Thus there exists an element $m \in M^Q$ such that $me = m$ and $\overline{m}kG = U$ where $\overline{m} \in M(Q)$ is the image of $m \in M^Q$ in $M(Q)$. We can take an element $x \in X^Q$ such that $f(x) = m$ and $xe = x$. Write $X = X_0 \oplus X_1$ where X_0 is a projective kG/Q-module and each indecomposable summand of X_1 has a vertex properly contained in Q. And write $\underline{x = x_0 + x_1 \text{ with}}\ x_i \in X_i$. Then $x_0e = x$, $x_1e = x_1$ and $x_1 \in \sum_{R \subsetneq Q} \mathrm{Tr}_{R,Q}(X_1^R)$. Thus $\overline{m} = \overline{f(x_0)}$ and $\overline{f(x_0)}kG \cong U$. As X_0 is a kG/Q-module and $x_0e = e$, x_0kG is a homomorphic image of $[Q]ekG$ and we can conclude that $x_0kG \cong [Q]ekG \cong U$. Set $V = x_0kG$. Then V is a direct summand of X_0 (and of X). Thus we have proved that we have a direct sum decomposition of kG-modules

$$X = V \oplus V'$$

such that $V \cong U$, $f(Q)(V(Q)) = U \subset M(Q)$ and $f(Q) \downarrow_{V(Q)}$: $V(Q) \to M(Q)$ induces isomorphisms

$$f(Q) \downarrow_{V(Q)} : \ V(Q) \to U$$

Write $Y = Y_0 \oplus Y_1$ where Y_0 is a projective kG/Q-module and each indecomposable summand of Y_1 has a vertex properly contained in Q. And write $g(m) = y_0 + y_1$ with $y_i \in Y_i$. Then $y_0e = y_0$ and $g(Q)(\overline{m}) = \overline{y_0} \in Y_0(Q)$. By the similar argument as above, it follows that $y_0kG \cong [Q]ekG \cong U$ and we have a direct sum decomposition of kG-modules

$$Y = W \oplus W'$$

such that $W \cong U$, $g(Q)(U) = W(Q) \subset Y(Q)$.

Let $\lambda : V \to X$, $\mu : Y \to W$ be the injection and projection with respect to the above decompositions and consider the maps $f' = f \circ \lambda : V \to M$ and $g' = \mu \circ g : M \to W$. Then $f'(Q) = f(Q) \circ \lambda(Q)$ and $g'(Q) = \mu(Q) \circ g(Q)$. By the discussions above, the composite $g'(Q) \circ f'(Q) : V(Q) \to M(Q) \to W(Q)$ is an isomorphism. As $(g' \circ f')(Q) = g'(Q) \circ f'(Q)$, it follows that the map $g' \circ f : V \to M \to W$ is an isomorphism and that V is isomorphic to a summand of M. □

Lemma 3.2. *Assume that $M \downarrow_Q$ is a permutation kQ-module and let $0 \to N \to X \to M \to 0$ be a Q-projective cover of M. Then $0 \to N(Q) \to X(Q) \to M(Q) \to 0$ is a minimal projective cover of a $kN_G(Q)/Q$-module $M(Q)$.*

Proof. As the sequence $0 \to N \to X \to M \to 0$ is Q-split, the resulting sequence $0 \to N(Q) \to X(Q) \to M(Q) \to 0$ is exact and a projective presentation of a $kN_G(Q)/Q$-module $M(Q)$. We also have that $N \downarrow_Q$ is a permutation module. By Lemma 3.1, $N(Q)$ has no projective $kN_G(Q)/Q$-summand and the lemma follows. □

The results in this section are obtained by joint works with Kunugi.

4. Finite Groups with Sylow p-Subgroup $M_{n+1}(p)$

Let p be an odd prime and $n \geqq 2$ be an integer. Consider the p-group $M_{n+1}(p) = P$ of order p^{n+1} given in Section 1. We use notations in Section 1. And set

$$P_0 = \langle\, y \,\rangle, \quad z ; = y^{p^{n-1}} = [x, y], \quad Z ; = \langle\, z \,\rangle$$

$$R = \langle\, x \,\rangle, \quad Z(P) = \langle\, y^p \,\rangle, \quad Q ; = C_P(R) = R \times Z(P)$$

For an integer i,

$$(y^i x)^p = y^{ip}$$

Thus

$$\Omega_p(P) = \langle\, x,\ z \,\rangle, \quad x \sim_P xz^i,\ 0 \leqq i \leqq p-1, \quad Z(P) = \langle\, y^p \,\rangle$$

and it follows that a nontrivial subgroup S of P contains Z or is conjugate to R in P.

Let G be a finite group with Sylow subgroup $P = M_{n+1}(p)$ such that there exists a normal subgroup G_0 satisfying that

$$G = R \ltimes G_0, \quad G_0 \cap P = P_0$$

Set

$$H = N_G(P_0) = R \ltimes N_{G_0}(P_0), \quad N_{G_0}(P_0) = H_0$$

Then $N_G(P) \subset N_G(P_0) = H$ and $H/O_{p'}(C_G(P))$ isomorphic to a subgroup of $\langle\, t_0 \,\rangle \ltimes P$. Set

$$e = |H/PC_G(P)|, \quad H = \langle\, t,\ PC_G(P) \,\rangle$$

so that $t^e \in O_{p'}(C_G(P))$ and $H/O_{p'}(H)$ is the group H in Section 1.

In this section, we shall be concerned with the principal block algebras $B_0(kG)$ and $B_0(kH)$ of kG and kH.
Notice that G and H have the same p-local structure.

4.1. p-Locals. Let M be a Broué-Puig indecomposable $k[G \times H]$-direct summand of of $B_0(kG)$ with vertex ΔP. As we are working on the principal block case, M is a Scott $k[G \times H]$-module with vertex ΔP. We investigate Brauer constructions $M(\Delta S)$ for nontrivial subgroups S of P.

4.1.1. Z. By a theorem of Burnside, $C_{G_0}(Z)$ is p-nilpotent and therefore so is $C_G(Z)$. In particular,

$$N_G(Z) = O_{p'}(C_G(Z))N_H(Z)$$

and $M(\Delta Z) = B_0(kC_G(Z)) = B_0(kC_H(Z))$.

4.1.2. $S \supset Z$. Let $S \subset P$ with $S \supset Z$. Then $N_G(S) \subset N_G(Z)$ because $Z \subset S \cap G_0$ and $S \cap G_0$ is cyclic.
Thus $M(\Delta S) = B_0(kC_G(S)) = B_0(kC_H(S))$.

4.1.3. R. We can see that

$$N_G(R) = R \ltimes N_{G_0}(R) = R \times C_{G_0}(R) = C_G(R)$$

Set $C = C_G(R)$ and $C_0 = C_{G_0}(R)$. Then $Q = R \times Z(P)$ is a Sylow p-subgroup of C and $Z(P)$ is a Sylow p-subgroup of C_0. We also have that

$$N_H(R) = C_H(R) = R \times C_{H_0}(R)$$

Set $K_0 = C_{H_0}(R)$. Then $N_{C_0}(Z) = K_0 O_{p'}(C_{C_0}(Z))$ by the following facts.

$$C_{H_0}(R) \subset N_{C_0}(Z(P)), \quad N_{C_0}(Z(P)) = C_{H_0}(R) O_{p'}(N_{C_0}(Z(P)))$$

$$N_{C_0}(Z(P)) \subset N_{C_0}(Z) = \langle\, t,\ C_{C_0}(Z)\,\rangle, \quad N_{C_0}(Z) = N_{C_0}(Z(P)) O_{p'}(C_{C_0}(Z))$$

As $(kC_0, kN_{C_0}(Z))$-module, $B_0(kC_0) = N' \oplus$ proj. where N' is a Broué-Puig module for $B_0(kC_0)$ and $B_0(kN_{C_0}(Z))$. Thus by the result above, as (kC_0, kK_0)-module,

$$B_0(kC_0) = N_0 \oplus \text{proj.}$$

where N_0 is indecomposable and has a vertex $\Delta Z(P)$ (Actually, in the situation here, $N_0 = N'$). N_0 gives a stable equivalence between $B_0(kC_0)$ and $B_0(kK_0)$. By a result of Rouquier, there exists a two terms Rickard complex $\mathbf{Y_0}$ for $B_0(kC_0)$ and $B_0(kK_0)$ of the following form,

$$\mathbf{Y_0}\ ;\ \cdots \to 0 \to Y_0 \xrightarrow{\nu_0} N_0 \to 0 \to \cdots$$

where Y_0 is a projective $k[C_0 \times K_0]$-module. If $Q_0 \xrightarrow{\nu_0'} N_0 \to 0$ is a projective cover of N_0, then Y_0 can be taken from a direct summand of Q_0 and $\nu_0 = \nu_0' \downarrow_{Y_0}$. We know that

$$B_0(kC) = B_0(kC_G(R)) = kR \otimes_k B_0(kC_0), \quad B_0(kC_H(R)) = kR \otimes_k B_0(kK_0)$$

Thus as $k[C \times C_H(R)]$-module,

$$B_0(kC) = N \oplus \Delta R\text{-proj.}$$

where N is a Broué-Puig module for $B_0(kC)$ and $B_0(kC_H(R))$. As $N_{G\times H}(\Delta R) = C_G(R) \times C_H(R)$, we have $M(\Delta R) = N$.
The complex $\mathbf{Y_0}$ can be lifted to a Rickard complex for $B_0(kC_G(R))$ and $B_0(kC_H(R))$ as follows. By the canonical epimorphism $\Delta R(C_0 \times K_0)/\Delta R \cong C_0 \times K_0$, the inflated complex $\widetilde{\mathbf{Y_0}}$ of $k[\Delta R(C_0 \times K_0)]$-modules of $\mathbf{Y_0}$ can be constructed.

$$\widetilde{\mathbf{Y_0}}\ ;\ \cdots \to 0 \to \widetilde{Y_0} \xrightarrow{\widetilde{\nu_0}} \widetilde{N_0} \to 0 \to \cdots$$

Then the induced complex $\mathbf{Y} = \widetilde{\mathbf{Y_0}} \uparrow^{C_G(R)\times C_H(R)}$ is the desired Rickard complex for $B_0(C_G(R))$ and $B_0(C_H(R))$. The degree 0 term of $\mathbf{Y}$ is $\widetilde{N_0} \uparrow^{C_G(R)\times C_H(R)}$ and is isomorphic to $N = M(\Delta R)$. Thus $\mathbf{Y}$ has the form

$$\mathbf{Y}\ ;\ \cdots \to 0 \to Y \xrightarrow{\nu} M(\Delta R) \to 0 \to \cdots$$

where $Y = \widetilde{Y_0} \uparrow^{C_G(R)\times C_H(R)}$.
Let $Q_0 \xrightarrow{\nu_0'} N_0 \to 0$ be a projective cover of N_0 as before so that $Q_0 = Y_0 \oplus Z_0$ for some projective $k[C_0 \times K_0]$-module and $\nu_0 = \nu_0' \downarrow_{Y_0}$. Set

$$Q = \widetilde{Q_0} \uparrow^{C_G(R)\times C_H(R)}, \quad Y = \widetilde{Y_0} \uparrow^{C_G(R)\times C_H(R)}, \quad Z = \widetilde{Z_0} \uparrow^{C_G(R)\times C_H(R)}$$

Then the resulting sequence $Q \xrightarrow{\nu'} N \to 0$ is a ΔR-projective cover of $N = M(\Delta R)$, $Q = Y \oplus Z$ and $\nu = \nu' \downarrow_Y$.

By our construction, each indecomposable summand of Y has a vertex ΔR. Let $X' \xrightarrow{\mu'} M \to 0$ be a ΔR-projective cover of M. Then its Brauer construction $X'(\Delta R) \to M(\Delta R) \to 0$ is a ΔR-projective cover of $kN_{G\times H}(\Delta R)$-module $M(\Delta R)$. Thus we have a decomposition $X' = X \oplus W$ of $k[G\times H]$-modules such that each indecomposable summand of X has a vertex ΔR and $X(\Delta R) = Y$. Now set $\mu = \mu' \downarrow_X$ and set

$$\mathbf{X}\ ;\ \cdots \to 0 \to X \xrightarrow{\mu} M \to 0 \to \cdots$$

Then by our construction we have $\mathbf{X}(\Delta R) = \mathbf{Y}$. And for $S \subset P$ with $S \supset Z$, we have $\mathbf{X}(\Delta S) = M(\Delta S)$.

Now a result of Rouquier says the following fact.

Lemma 4.1. *The complex $\mathbf{X}$ induces a stable equivalence of Rickard type between $B_0(kG)$ and $B_0(kH)$.*

4.1.4. *Stable Equivalence.* Let W be the $(B_0(kG), B_0(kH))$-bimodule given in the previous subsections. And let $P \xrightarrow{\lambda'} W \to 0$ be a projective cover of W so that we have an exact sequence of $(B_0(kG), B_0(kH))$-bimodule

$$X \oplus P \xrightarrow{\lambda} M \to 0 \text{ (exact)}$$

where $\lambda = (\mu, \nu \circ \lambda')$ with $\nu = \mu' \downarrow_W$. Set $M_0 = \Omega^{-1}(\operatorname{Ker}\lambda)$ so that we have an exact sequence of $(B_0(kG), B_0(kH))$-modules of the form

$$0 \to X \to M \oplus P_0 \to M_0 \to 0$$

where P_0 is a projective $(B_0(kG), B_0(kH))$-bimodule.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\lambda \arrow[r] \arrow[d, equal] & X \oplus P \arrow[r, "\lambda"] \arrow[d, "f_1"'] & M \arrow[r] \arrow[d, "f_0"'] & 0 \\
0 \arrow[r] & \operatorname{Ker}\lambda \arrow[r] & P_0 \oplus P \arrow[r] & M_0 \arrow[r] & 0
\end{tikzcd}$$

Lemma 4.2. *The functor $- \otimes_{B_0(kG)} M_0 : \text{mod-}B_0(kG) \to \text{mod-}B_0(kH)$ gives a stable equivalence of Morita type between $B_0(kG)$ and $B_0(kH)$.*

Set $A = B_0(kG)$ and $B = B_0(kH)$.

For a nonprojective indecomposable A-module V, Let $F(V)$ be a nonprojective B-summand of $V \otimes_A M_0$ so that $F(V)$ is indecomposable and $V \otimes_A M_0 = F(V) \oplus \text{proj}$.

Assume that $V \downarrow_R$ is a permutation kR-module and is not R-projective. Notice that a kC_0-module $V(R)$ has no projective summand by Lemma 3.1. Assume, furthermore that $V(R)$ is simple. Then a B-module $F(V)$ is obtained by the following way.

Set $A_0 = B_0(kC_G(R)/R) = B_0(kC_0)$ and $B_0 = B_0(kK_0)$. Then by a result of Puig-Rickard [11],

$$(V \otimes_A M)(R) \cong V(R) \otimes_{A_0} M(\Delta R) = V(R) \otimes_{A_0} N_0$$

and

$$(V \otimes_A X)(R) \cong V(R) \otimes_{A_0} X(\Delta R) = V(R) \otimes_{A_0} Y_0$$

as $B_0(kK_0)$-modules. Thus

$$(V \otimes_A \mathbf{X})(R) \cong V(R) \otimes_{A_0} \mathbf{Y_0}$$

By the discussion on p-locals, we can write $V \otimes_A M = F'(V) \oplus U'$ where $F'(V)$ is indecomposable and U' is R-projective. Then

$$F'(V)(R) \oplus U'(R) = (V \otimes_A M)(R) \cong V(R) \otimes_{A_0} N_0$$

As we are assuming that $V(R)$ is simple, $V(R) \otimes_{A_0} N_0$ is indecomposabe. In particular, $U'(R) = 0$ and U' is projective. If we set $U = V(R) \otimes_{A_0} N_0$, then by a result of Rouquier [13, 14], one of the following occurs.

$$V(R) \otimes_{A_0} \mathbf{Y_0} \ : \quad \cdots \to 0 \to \quad 0 \to \quad U \to 0 \to \cdots \qquad (*.1)$$

$$V(R) \otimes_{A_0} \mathbf{Y_0} \ : \quad \cdots \to 0 \to \quad Q(U) \xrightarrow{\rho} \quad U \to 0 \to \cdots \qquad (*.2)$$

where $Q(U) \xrightarrow{\rho} U \to 0$ is a projective cover of a $B_0(kK_0)$-module U.

We have proved the following lemma.

Lemma 4.3. *Let V be an indecomposable $B_0(kG)$-module such that $V \downarrow_R$ is a permutation kR-module and is not R-projective. Assume, furthermore that $V(R)$ is simple. Then $F(V) = V \otimes_{B_0(kG)} M$ or $F(V) = \Omega^{-1}\Omega_R(V \otimes_{B_0(kG)} M)$ according to the case $(*.1)$ occurs or the case $(*.2)$ occurs.*

Corollary 4.4. *Let V be a $B_0(kG)$-module satisfying the conditions in the Lemma and assume that $\mathrm{Hom}_k(V, k) \otimes_k V = k_G \oplus V'$ for some R-projective kG-module V'.*

Then

$$\mathrm{Hom}_k(F(V), k) \otimes_k F(V) = k_H \oplus V_0$$

for some projective kH-module V_0. In particular, $F(V) \downarrow_P$ is an endo-trivial kP-module.

Proof. By our construction of the functor F, we can write

$$\mathrm{Hom}_k(F(V), k) \otimes_k F(V) = k_H \oplus V_0$$

where V_0 is an R-projective p-permutation kH-module. Thus it suffices to show that $V_0(R) = 0$. We use the notations in the discussion before the lemma.

By a result of Puig-Richard,

$$k_{C_0} \oplus V_0(R) = (\mathrm{Hom}_k(V, k) \otimes_k V)(R) \cong \mathrm{Hom}_k(V(R), k) \otimes_k V(R) \qquad (*)$$

as kC_0-modules. Then for $U = V(R) \otimes_{A_0} N_0$,

$$k_{C_0} \oplus U_0 \cong \mathrm{Hom}_k(U, k) \otimes_k U$$

as kC_0-modules where U_0 is a projective kK_0-module. As $kC_H(R)$-modules, we also have

$$k_{C_H(R)} \oplus U_1 \cong \mathrm{Hom}_k(\Omega_R(U), k) \otimes_k \Omega_R(U)$$

where U_1 is an R-projective $kC_H(R)$-module. By $(*)$, we can see that a source of $B_0(kC_0)$-module $V(R)$ is $k_{Z(P)}$ or $\Omega(k_{Z(P)})$. By properties of Roquier's complex $\mathbf{Y_0}$, the case $(*.1)$ occurs if a source of $V(R)$ is $k_{Z(P)}$ and the case $(*.2)$ occurs if a source of $V(R)$ is $\Omega(k_{Z(P)})$.

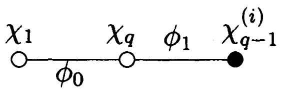
If the case $(*.1)$ occurs, then U is a simple $B_0(kC_H(R))$-module. If the case $(*.2)$ occurs, then an R-projective cover $\Omega_R(U)$ of U as $kC_H(R)$-module is a simple $B_0(kC_H(R))$-module. Notice that simple $B_0(kC_H(R))$-modules are one dimensional. Thus $U_0 = 0$ in the case $(*.1)$ and $U_1 = 0$ in the case $(*.2)$. □

5. Examples

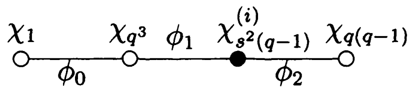
We shall give some examples of groups G with Sylow p-subgroup $M_{n+1}(p)$ where we could check that simple $B_0(kG)$-modules V satisfy the assumption in Lemma 4.3.

Our groups G are constructed from G_0 isomorphic to $SL(2,q)$, $SU(3,q^2)$ and $Sp(4,q)$ for suitably chosen prime power q such that $p \mid q+1$, $p \mid q^2-q+1$ and $p \mid q^2+1$, respectively. These groups G_0 have cyclic Sylow p-subgroups and the Brauer trees of $B_0(kG_0)$ are the following shapes. In the figures, χ_k is an ordinary irreducible characters of degree k. See the paper by Fong and Srinivasan [3].

$SL(2,q)$, $p \mid q+1$

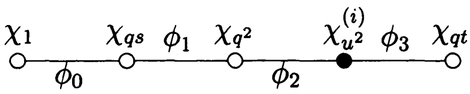


$SU(3,q^2)$, $p \mid q^2-q+1$



$(s = q+1)$

$Sp(4,q)$, $p \mid q^2+1$



$(s = \frac{1}{2}(q+1)^2,\ t = \frac{1}{2}(q-1)^2,\ u = (q^2-1))$

5.1. $SL(2,q)$. Let r be a prime power and p be an odd prime such that p divides $r+1$. Write $r+1 = p^{n-1}\ell'$, $(p,\ell') = 1, n \geqq 2$. Set $q = r^p$. Then $q+1 = p^n\ell$ for some positive integer ℓ with $(p,\ell) = 1$.
Set

$$G_0 = SL(2,q),\ C_0 = SL(2,r).\quad R = \mathcal{G}(GF(q)/GF(r)) = \langle\, x \,\rangle,\quad G = R \ltimes G_0$$

Let $B_0 = T_0 \ltimes U_0$ be a Borel subgroup of G_0 where $|T_0| = q-1$ and $|U_0| = q$. We have an R-invariant subgroup $F_0 \supset Z(G_0)$ of order $q+1$ such that $F_0 \cap C_0$ is of order $r+1$ and $B_0 \cap F_0 = Z(G_0)$.
Let $P_0 \subset F_0$ be a Sylow p-subgroup of G_0 and set $P = R \ltimes P_0$. We have that $P \cong M_{n+1}(p)$.

$B_0(kG_0)$ and $B_0(kC_0)$ have two simple modules

$$B_0(kG_0) : \phi_0 = k_{G_0},\quad \phi_1,\quad \dim_k \phi_1 = q-1,\quad B_0(kC_0) : \theta_0 = k_{C_0},\quad \theta_1,\quad \dim_k \theta_1 = r-1$$

A simple module ϕ_1 is the heart of a projective cover $P(\phi_0) = P(k_{G_0})$.

$$P(\phi_0) = k_{B_0} \uparrow^{G_0} \quad \text{and is uniserial of the form} \quad P(\phi_0) = \begin{matrix} \phi_0 \\ \phi_1 \\ \phi_0 \end{matrix}$$

The entirely same thing occurs for a projective cover $Q(\theta_0)$ of θ_0.

Set $B = R \ltimes B_0$ and $P_R(k_G) = k_B \uparrow^G$. $P_R(k_G)$ is an extension of $P(k_G)$ and therefore is uniserial of length 3 with form

$$P_R(k_G) = \begin{matrix} \varphi_0 \\ \varphi_1 \\ \varphi_0 \end{matrix}$$

where $\varphi_0 = k_G$ and $\varphi_1 \downarrow_{G_0} = \phi_1$. It is not hard to see that $\varphi_1 \downarrow_R$ is a permutation kR-module and

$$P_R(k_G)(R) = Q(k_{C_0}), \quad \varphi_1(R) = \theta_1$$

5.2. $SU(3, q^2)$**.** Let r be a prime power and p be a prime with $p \geqq 5$ such that p divides $r^2 - r + 1$. Write $r^2 - r + 1 = p^{n-1}\ell'$, $(p, \ell') = 1, n \geqq 2$. Set $q = r^p$. Then $q^2 - q + 1 = p^n \ell$ for some positive integer ℓ with $(p, \ell) = 1$.
Set

$$G_0 = SU(3, q^2),\ C_0 = SU(3, r^2). \quad R = \mathcal{G}(GF(q)/GF(r)) = \langle\, x \,\rangle, \quad G = R \ltimes G_0$$

Let $B_0 = T_0 \ltimes U_0$ be a Borel subgroup of G_0 where $|T_0| = (q+1)(q-1)$ and $|U_0| = q^3$. We have an R-invariant subgroup $F_0 \supset Z(G_0)$ of order $q^2 - q + 1$ such that $F_0 \cap C_0$ is of order $r^2 - r + 1$ and $B_0 \cap F_0 = Z(G_0)$.
Let $P_0 \subset F_0$ be a Sylow p-subgroup of G_0 and set $P = R \ltimes P_0$. We have that $P \cong M_{n+1}(p)$.

$B_0(kG_0)$ and $B_0(kC_0)$ have three simple modules

$$B_0(kG_0)\ :\ \phi_0 = k_{G_0}, \quad \phi_1, \quad \dim_k \phi_1 = q^3 - 1, \quad \phi_2, \quad \dim_k \phi_2 = q(q-1)$$

$$B_0(kC_0)\ :\ \theta_0 = k_{C_0}, \quad \theta_1, \quad \dim_k \theta_1 = r^3 - 1, \quad \theta_1, \quad \dim_k \theta_1 = r(r-1)$$

Simple modules ϕ_1 and ϕ_2 are described as follows.

5.2.1. ϕ_1. A simple module ϕ_1 is the heart of a projective cover $P(\phi_0) = P(k_{G_0})$.

$$P(\phi_0) = k_{B_0} \uparrow^{G_0} \quad \text{and is uniserial of the form} \quad P(\phi_0) = \begin{matrix} \phi_0 \\ \phi_1 \\ \phi_0 \end{matrix}$$

The same thing occurs for a projective cover $Q(\theta_0)$ of θ_0.

Set $B = R \ltimes B_0$ and $P_R(k_G) = k_B \uparrow^G$. $P_R(k_G)$ is an extension of $P(k_G)$ and therefore is uniserial of length 3 with form

$$P_R(k_G) = \begin{matrix} \varphi_0 \\ \varphi_1 \\ \varphi_0 \end{matrix}$$

where $\varphi_0 = k_G$ and $\varphi_1 \downarrow_{G_0} = \phi_1$. It is not hard to see that $\varphi_1 \downarrow_R$ is a permutation kR-module and

$$P_R(k_G)(R) = Q(k_{C_0}), \quad \varphi_1(R) = \theta_1$$

5.2.2. ϕ_2. Set $B = R \ltimes B_0$. By the knowledge of the character tables of G_0 and B_0, we can see that there exists a simple $B_0(kG_0)$-module ϕ_2 of dimension $q(q-1)$ and the restriction $\phi' = \phi_2 \downarrow_{B_0}$ is a simple kB_0-module which is R-invariant. $\phi' \downarrow_{Z(U_0)}$ does not contain $k_{Z(U_0)}$. The block of kB which covers ϕ' has a cyclic defect group R and B_0 is a p'-group. Thus Alperin-Brauer-Dade-Glauberman theory can be applied. Notice that $C_{B_0}(R)$ is a Borel subgroup of $C_0 = SU(3, r^2)$.

ϕ' has a unique extension φ' to B as $|B : B_0|$ is a p-group. The Brauer-Glauberman correspondent θ' of ϕ' does not contain $Z(C_{U_0}(R))$ in its kernel. Thus $\dim_k \theta' = r(r-1)$.

We see that $\dim_k \phi' - \dim_k \theta' = q(q-1) - r(r-1) \equiv 0 \pmod p$. Thus the extension φ' of ϕ' has a trivial source module and

$$\varphi'(R) = \theta'$$

ϕ_2 also has a unique extension φ_2 to G. Then $\varphi_2 \downarrow_B$ is an extension of ϕ'. Thus $\varphi_2 \downarrow_B = \varphi'$. $\varphi_2(R)$ is a $B_0(kC_0)$-module and $\varphi_2(R) \downarrow_{B \cap C_0} = \theta'$. Such a $B_0(kC_0)$-module must be simple and

$$\varphi_2(R) = \theta_2$$

The simple $B_0(kG_0)$-module ϕ_2 is self-dual and we can see that $\phi_2 \otimes \phi_2 = k_{G_0} \oplus$ defect 0 blocks. Thus

$$\varphi_2^* \otimes \varphi_2 = k_G \oplus \ R\text{-projective}$$

5.3. $Sp(4, q)$. Let r be a prime power and p be a prime with $p \geqq 5$ such that p divides $r^2 + 1$. Write $r^2 + 1 = p^{n-1}\ell'$, $(p, \ell') = 1, n \geqq 2$. Set $q = r^p$. Then $q^2 + 1 = p^n \ell$ for some positive integer ℓ with $(p, \ell) = 1$.
Set

$$G_0 = Sp(4, q),\ C_0 = Sp(4, r).\quad R = \mathcal{G}(GF(q)/GF(r)) = \langle\ x\ \rangle, \quad G = R \ltimes G_0$$

Let $B_0 = T_0 \ltimes U_0$ be a Borel subgroup of G_0 where $|T_0| = (q-1)^2$ and $|U_0| = q^4$. Let $W = N_G(T)/T = \langle\ w_a,\ w_b\ \rangle$ be the Weyl group of G where w_a is a reflection corresponding to a long root. We have an R-invariant subgroup F_0 of order $q^2 + 1$ such that $F_0 \cap C_0$ is of order $r^2 + 1$ and $B_0 \cap F_0 = Z(G_0)$.
Let $P_0 \subset F_0$ be a Sylow p-subgroup of G_0 and set $P = R \ltimes P_0$. We have that $P \cong M_{n+1}(p)$.
$B_0(kG_0)$ and $B_0(kC_0)$ have four simple modules

$$\begin{aligned} B_0(kG_0)\ &:\ \phi_0 = k_{G_0}, \quad \phi_1,\ \dim_k \phi_1 = \frac{1}{2}q(q+1)^2 - 1, \\ &\ \ \phi_2,\ \dim_k \phi_2 = q^4 - \frac{1}{2}q(q+1)^2 + 1, \quad \phi_3,\ \dim_k \phi_3 = \frac{1}{2}q(q-1)^2 \\ B_0(kC_0)\ &:\ \theta_0 = k_{C_0}, \quad \theta_1, \quad \dim_k \theta_1 = \frac{1}{2}r(r+1)^2 - 1, \\ &\ \ \theta_2, \quad \dim_k \theta_2 = r^4 - \frac{1}{2}r(r+1)^2 + 1, \quad \theta_3, \quad \dim_k \theta_3 = \frac{1}{2}r(r-1)^2 \end{aligned}$$

Simple modules ϕ_1, ϕ_2 and ϕ_3 are described as follows.

5.3.1. ϕ_1, ϕ_2. Let $B_0 \subset K_0 = \langle\, w_a,\ B_0 \,\rangle = L_0 \ltimes V_0$ be a maximal parabolic subgroup of G_0. A simple module ϕ_1 is the heart of a projective cover $P(\phi_0) = P(k_{G_0})$. We have

$$k_{K_0} \uparrow^{G_0} = P(k_{G_0}) \oplus P_0'$$

where P' is a simple projective kG_0-module of dimension $\frac{1}{2}q(q^2+1)$. $P(\phi_0)$ is uniserial of the form

$$P(\phi_0) = \begin{matrix} \phi_0 \\ \phi_1 \\ \phi_0 \end{matrix}$$

The same thing occurs for a projective cover $Q(\theta_0)$ of θ_0.

Set $K = R \ltimes K_0$. Then

$$k_K \uparrow^G = P_R(k_G) \oplus P'$$

where $P_R(k_G)$ is an extension of $P(k_G)$. In particular, $P_R(k_G)$ is uniserial of length 3 with form

$$P_R(k_G) = \begin{matrix} \varphi_0 \\ \varphi_1 \\ \varphi_0 \end{matrix}$$

where $\varphi_0 = k_G$ and $\varphi_1 \downarrow_{G_0} = \phi_1$. It is not hard to see that $\varphi_1 \downarrow_R$ is a permutation kR-module and

$$P_R(k_G)(R) = Q(k_{C_0}), \quad \varphi_1(R) = \theta_1$$

Write $k_{B_0} \uparrow^{K_0} = k_{K_0} \oplus \rho_0$. ρ_0 is the Steinberg module of $K_0/V_0 = L_0 = GL(2,q)$. We have

$$\rho_0 \uparrow^{G_0} = P(\phi_1) \oplus P_0''$$

where P_0'' is a simple projective kG_0-module of dimension $\frac{1}{2}q(q^2+1)$. $P(\phi_1)$ has the form

$$P(\phi_1) = \begin{matrix} \phi_1 \\ \phi_0 \oplus \phi_2 \\ \phi_1 \end{matrix}$$

for some simple kG_0-module ϕ_2. The same thing occurs for a projective cover $Q(\theta_1)$ of θ_1 and we have a simple kC_0-module θ_2.

It is not hard to see that ρ_0 has a unique extension ρ to K and ρ is a p-permutaion module. And we have

$$\rho \uparrow^G = P_R(\phi_1) \oplus P''$$

where $P_R(\phi_1)$ is an extension of $P(\phi_1)$. In particular, $P_R(\phi_1)$ has the form

$$P_R(\phi_1) = \begin{matrix} \varphi_1 \\ \varphi_0 \oplus \varphi_2 \\ \varphi_0 \end{matrix}$$

where $\varphi_2 = \phi_2$. It is not hard to see that $\varphi_2 \downarrow_R$ is a permutation kR-module and

$$P_R(\phi_1)(R) = Q(\theta_1), \quad \varphi_2(R) = \theta_2$$

5.3.2. ϕ_3. By the knowledge of the character tables of G_0 and K_0, we can see that there exists a simple $B_0(kG_0)$-module ϕ_3 of dimension $\frac{1}{2}q(q-1)^2$ and the restriction $\phi' = \phi_3 \downarrow_{K_0}$ is a simple kK_0-module which is R-invariant. $\phi' \downarrow_{Z(U_0)}$ does not contain $k_{Z(U_0)}$. The block of kK which covers ϕ' has a cyclic defect group R and K_0 is a p'-group. Thus Alperin-Brauer-Dade-Glauberman theory can be applied. Notice that $C_{K_0}(R)$ is a maximal parabolic subgroup of $C_0 = Sp(4,r)$).

ϕ' has a unique extension φ' to K as $|K : K_0|$ is a p-group. The Brauer-Glauberman correspondent θ' of ϕ' does not contain $Z(C_{U_0}(R))$ in its kernel. Thus $\dim_k \theta' = \frac{1}{2}r(r-1)^2$.

We see that $\dim_k \phi' - \dim_k \theta' = \frac{1}{2}(q(q-1)^2 - r(r-1)^2) \equiv 0 \pmod p$. Thus the extension φ' of ϕ' has a trivial source module and

$$\varphi'(R) = \theta'$$

ϕ_3 also has a unique extension φ_3 to G. Then $\varphi_3 \downarrow_K$ is an extension of ϕ'. Thus $\varphi_3 \downarrow_K = \varphi'$. $\varphi_3(R)$ is a $B_0(kC_0)$-module and $\varphi_3(R) \downarrow_{K\cap C_0} = \theta'$. Such a $B_0(kC_0)$-module must be simple and

$$\varphi_3(R) = \theta_3$$

The simple $B_0(kG_0)$-module ϕ_3 is self-dual and we can see that

$$\phi_3 \otimes \phi_3 = k_{G_0} \oplus \text{ defect 0 blocks}$$

Thus

$$\varphi_3^* \otimes \varphi_3 = k_G \oplus \ R\text{-projective}$$

References

[1] M.Broué, *Blocs,isométries parfaites, catégories dérivées* , C. R. Acad. Sci. Paris, 307-I, 13–18, 1988
[2] M.Broué, *Isométries parfaites, types de blocs, catégories dérivées* , Astérisque, 181–182, 61–92, 1990
[3] P.Fong and B.Srinivasan, *Brauer trees in classical groups*, J. Alg.,131, 179–225, 1990
[4] M.Holloway, S.Koshitani and N.Kunugi, *Blocks with nonabelian defect groups which have cyclic subgroups of index p*, Archiv der Mathematik 94, 101–116, 2010
[5] S. König and A. Zimmermann, Derived Equivalences for Group Ring, Springer Lecture Notes in Mathematics, 1685,1998
[6] M. Linckelmann, *Stable equivalences of Morita type for self-injective algebras and p-groups*, Math. Z., 223, 87–100,1996
[7] T.Okuyama and H.Sasaki, *Periodic Modules of Large Periods for Metacyclic p-Groups*, J. Alg.,144, 8–23, 1991
[8] J.Rickard, *Morita theory for derived categories*, Journal of London Math. Soc. (2) 39, 436–456, 1989 Math. Z., 223, 87–100,1996
[9] J.Rickard, *Derived categories and stable equiivalence*, Journal of Pure and Appl. Algebra, 61, 303–317, 1989
[10] J.Rickard, *Derived equivalences and derived functors*, Journal of London Math. Soc. (2) 43, 37–48, 1991
[11] J.Rickard, *Splendid equivalences: Derived categories and permutation modules*, Proc. London Math. Soc.(3), 72, 331–358, 1996
[12] J.Rickard, *Equivalences of derived categories for symmetric algebras*, Journal of Algebra, 257, 460–481, 2002

[13] R.Rouquier, *From stable equivalences to Rickard equivalences for blocks with cyclic defect*, Groups '93 Galway/St Andrews II(eds. C.M.Campbell *et al.*), London Mathematical Society Lecture Note Series, 212, 512–523, 1995

[14] R.Rouquier, *The derived category of blocks with cyclic defect groups*, in Derived Equivalences for Group Rings (S.König and A.Zimmermann), Springer Lecture Notes in Mathematics 1685, 1998

[15] R.Rouquier, *Block theory via stable and Rickard equivalences* , in Modular Representation Theory of Finite Groups (eds. M.J.Collins, B.J. Parshall and L.L.Scott), de Gruyter, 2001